Subexponential asymptotics of the stationary distributions of M/G/1-type Markov chains

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Abstract

This paper studies the subexponential asymptotics of the stationary distribution of an M/G/1-type Markov chain. We provide a sufficient condition for the subexponentiality of the stationary distribution. The sufficient condition requires only the subexponential integrated tail of level increments. On the other hand, the previous studies assume the subexponentiality of level increments themselves and/or the aperiodicity of the G-matrix. Therefore, our sufficient condition is weaker than the existing ones. We also mention some errors in the literature.

Keywords: Queueing, Subexponential asymptotics, M/G/1-type Markov chain, Periodicity, G-matrix, BMAP

1. Introduction

This paper considers an irreducible and positive-recurrent Markov chain \{(X_n, S_n); n \in \mathbb{Z}_+\} of M/G/1 type (Neuts 1989), where \(\mathbb{Z}_+ = \{0, 1, \ldots\}\). For each \(n \in \mathbb{Z}_+\), \(X_n\) takes values in \(\mathbb{Z}_+\). If \(X_n = 0\), \(S_n\) takes values in \(\mathbb{M}_0 \triangleq \{1, 2, \ldots, M_0\}\); and otherwise in \(\mathbb{M} \triangleq \{1, 2, \ldots, M\}\), where \(M_0\) and \(M\) are positive integers. The sets of states \{(0, j); j \in \mathbb{M}_0\} and \{(k, j); j \in \mathbb{M}\} \((k \in \mathbb{N} \triangleq \mathbb{Z}_+ \setminus \{0\})\) are called level 0 and level \(k\), respectively. Arranging the states in lexicographical order, the transition probability matrix \(T\) of
\((X_n, S_n)\) is given by
\[
T = \begin{pmatrix}
B(0) & B(1) & B(2) & B(3) & \cdots \\
C(0) & A(1) & A(2) & A(3) & \cdots \\
O & A(0) & A(1) & A(2) & \cdots \\
& O & A(0) & A(1) & \cdots \\
& & & & \ddots
\end{pmatrix},
\]
where \(A(k) \ (k \in \mathbb{Z}_+)\) is an \(M \times M\) matrix, \(B(0)\) is an \(M_0 \times M_0\) matrix, \(B(k) \ (k \in \mathbb{N})\) is an \(M_0 \times M\) matrix, and \(C(0)\) is an \(M \times M_0\) matrix. Let \(A = \sum_{k=0}^{\infty} A(k)\) and \(B = \sum_{k=1}^{\infty} B(k)\). It is clear that if \(T\) is stochastic, \(A\) is stochastic and \(B(0)e^{[M_0]} + Be^{[M]} = e^{[M_0]}\), where \(e^{[m]} \ (m \in \mathbb{N})\) denotes an \(m \times 1\) column vector whose elements are all equal to one (hereafter we may write simply \(e\) for \(e^{[m]}\) when its size is obvious).

Throughout this paper, we make the following assumption.

**Assumption 1.1** (a) \(T\) is stochastic and irreducible; (b) \(A\) is irreducible; (c) \(\rho \triangleq \pi \beta_A < 1\), where \(\pi\) is the stationary probability vector of \(A\) and \(\beta_A = \sum_{k=1}^{\infty} k A(k) e\); and (d) \(\beta_B \triangleq \sum_{k=1}^{\infty} k B(k) e\) is a finite vector.

It is known that under Assumption 1.1, the Markov chain \((X_n, S_n)\) is irreducible and positive recurrent (see Asmussen 2003, Chapter XI, Proposition 3.1). Thus \(T\) has a unique stationary probability vector \(x > 0\). Let \(x(k) \ (k \in \mathbb{Z}_+)\) denote a subvector of \(x\) corresponding to level \(k\). We then have \(x = (x(0), x(1), x(2), \ldots)\) and

\[
x(k) = x(0) B(k) + \sum_{l=1}^{k+1} x(l) A(k+1 - l), \quad k \in \mathbb{N}.
\]

We also define \(\bar{x}(k) = \sum_{l=k+1}^{\infty} x(l)\), which is positive for all \(k \in \mathbb{Z}_+\).

As is well known, M/G/1-type Markov chains arise from various queueing models with batch Markovian arrival processes (BMAPs) such as BMAP/PH/c queues; and BMAP/GI/1 queues with/without vacations, interruptions and priorities. However, it is difficult to obtain an explicit expression of \(\{x(k)\}\) and thus \(\{\bar{x}(k)\}\) of the M/G/1-type Markov chain, in general. They have to be computed by a recursive algorithm (see Neuts 1989; Ramaswami 1988; Schellhaas 1990), which implies that we are forced to perform a number of numerical experiments with various parameter sets in order to investigate the qualitative behavior of the stationary distribution. Therefore it is
a hot topic to study the tail asymptotics of the stationary distributions of structured Markov chains such as quasi-birth-and-death processes (QBDs), M/G/1-type Markov chains and more general GI/G/1-type ones. Especially, the light-tailed asymptotics has been studied by many researchers (Abate et al. 1994; Falkenberg 1994; Kimura et al. 2010; Li and Zhao 2005b; Li et al. 2007; Miyazawa 2004; Miyazawa and Zhao 2004; Möller 2001; Tai 2009; Takine 2004).

On the other hand, a smaller number of researchers have studied the subexponential asymptotics of the stationary distributions of GI/G/1-type Markov chains (including M/G/1-type ones). Asmussen and Möller (1999) and Li and Zhao (2005a) consider GI/G/1-type Markov chains having subexponential level increments. In the context of this paper, the subexponentiality of level increments means that $Y$ is subexponential (i.e., $Y \in \mathcal{S}$; see Definition A.2), where $Y$ is a random variable in $\mathbb{Z}_+$ such that

$$
\lim_{k \to \infty} \sum_{l=k+1}^{\infty} A(l) \frac{P(Y > k)}{P(Y > k)} = C_1 \geq O,
\lim_{k \to \infty} \sum_{l=k+1}^{\infty} B(l) \frac{P(Y > k)}{P(Y > k)} = C_2 \geq O,
$$

with $C_1 \neq O$ or $C_2 \neq O$.

In addition to $Y \in \mathcal{S}$, Asmussen and Möller (1999) and Li and Zhao (2005a) assume $Y_e \in \mathcal{S}$, where $Y_e$ denotes the discrete equilibrium random variable of $Y$, distributed with $P(Y_e = k) = P(Y > k)/E[Y]$ ($k \in \mathbb{Z}_+$). It should be noted that $Y \in \mathcal{S}$ does not necessarily imply $Y_e \in \mathcal{S}$ and vice versa (see Sigman 1999, Remark 3.5). Under these conditions, they show that for some $c > 0$,

$$
\lim_{k \to \infty} \frac{\varphi(k)}{P(Y_e > k)} = c, \quad Y_e \in \mathcal{S}.
$$

Takine (2004) derives the subexponential asymptotic formula (1) for an M/G/1-type Markov chain, assuming $Y_e \in \mathcal{S}$ but not $Y \in \mathcal{S}$. The result implies that the subexponential asymptotic formula (1) does not necessarily require $Y \in \mathcal{S}$, i.e., the subexponentiality of level increments. However, the proof of (1) given in Takine (2004) requires the aperiodicity of the $G$-matrix.

In this paper, we show that (1) holds without an additional condition such as $Y \in \mathcal{S}$ and the aperiodicity of the $G$-matrix. Therefore, our sufficient condition for the subexponentiality of the stationary distribution is weaker than those presented in the literature (Asmussen and Möller 1999; Li and Zhao 2005a; Takine 2004), though our result is limited to the M/G/1-type Markov chain. We also point out that some asymptotic formulae given in Li
and Zhao (2005a) are incorrect. In fact, the formulae include the inverse of a singular matrix and thus are inconsistent with our results.

The rest of this paper is divided into two sections. In Section 2, we provide some preliminary results of M/G/1-type Markov chains, and present our main results in Section 3.

2. Preliminaries

Throughout this paper, we use the following conventions. Let \( Z \) denote the set of all integer numbers, i.e., \( Z = \{0, \pm 1, \pm 2, \ldots\} \). For any set \( A \), let \( |A| \) denote the cardinality of \( A \). For any random variable \( X \) in \( \mathbb{Z}_+ \) with finite positive mean, let \( X_e \) denote the discrete equilibrium random variable of \( X \) such that \( P(X_e \leq k) = \sum_{l=0}^{k} P(X > l) / E[X] \) \((k \in \mathbb{Z}_+)\). The superscript “\( t \)” represents the transpose operator for vectors and matrices.

Let \( I \) denote the identity matrix. For any matrix \( X \), \([X]_{i,j}\) represents the \((i,j)\)th element of \( X \). For any summable matrix sequence \( \{M(k); k \in \mathbb{Z}_+\} \), let \( M(k) = \sum_{l=k+1}^{\infty} M(l) \) \((k \in \mathbb{Z}_+)\). Finally, if a sequence of nonnegative matrices \( \{M(k); k \in \mathbb{Z}_+\} \) satisfies \( \lim_{k \to \infty} M(k)/f(k) = C \geq O \) for some \( \{f(k) > 0; k \in \mathbb{Z}_+\} \), then we write \( M(k) \sim C f(k) \). The conventions for matrices are also applied to vectors and scalars in an appropriate manner.

Let \( G \) denote an \( M \times M \) matrix such that \([G]_{i,j} = P(S_{\nu(k)} = j \mid X_0 = k + 1, S_0 = i)\) for any given \( k \) \((k \in \mathbb{N})\), where \( \nu(k) = \inf\{n \in \mathbb{N}; X_n = k, X_l > k \ (l = 0, 1, \ldots, n-1)\} \). It is known (Neuts 1989) that \( G \) is the minimal nonnegative solution of

\[
G = \sum_{k=0}^{\infty} A(k) G^k.
\]

If Assumption 1.1 (b) and (c) hold, \( G \) is stochastic (see Neuts 1989, Theorem 2.3.1).

**Proposition 2.1 (Kimura et al. 2010, Proposition 2.1)** Suppose Assumption 1.1 (a) and (b) hold. Then \( G \) has only one irreducible class, which is denoted by \( M_\bullet \). In some cases, \( M_\bullet = M \), i.e., \( M_T \triangleq M \setminus M_\bullet = \emptyset \). Furthermore, by a permutation, \( G \) takes a form:

\[
G = M_\bullet \begin{pmatrix} M_\bullet & M_T \\ M_T & G_0 \end{pmatrix},
\]

(2)
where \( G_{\bullet} \) is irreducible, \( G_T \) (if any) is strictly lower-triangular and \( G_o \) (if any) is a nonnegative matrix such that \( G_o e \leq e \).

In what follows, the states in \( \mathbb{M} \) are arranged in such a way that \( G \) takes the form of (2). Proposition 2.1 shows that \( G \) has a unique stationary probability vector, which is denoted by \( g \) hereafter (i.e., \( gG = g \) and \( ge = 1 \)). Note that \( g \) has positive elements in the positions corresponding to \( \mathbb{M}_{\bullet} \) and the other elements (if any) are all equal to zero.

Let \( R(k) \) and \( R_0(k) \) \((k \in \mathbb{N})\) denote
\[
R(k) = \sum_{m=0}^{\infty} A(k + m + 1)G^m(I - U(0))^{-1}, \quad k \in \mathbb{N},
\]
\[
R_0(k) = \sum_{m=0}^{\infty} B(k + m)G^m(I - U(0))^{-1}, \quad k \in \mathbb{N},
\]
respectively, where
\[
U(0) = \sum_{m=0}^{\infty} A(m + 1)G^m.
\]
For convenience, let \( R(0) = O \) and \( R_0(0) = O \). We then have (see Ramaswami 1988)
\[
x(k) = x(0)R_0(k) + \sum_{j=1}^{k} x(j)R(k - j), \quad k \in \mathbb{N}.
\]
We also have the following results (see Takine 2003, Lemma 14; Takine 2004, Lemma 3).

**Proposition 2.2** Suppose Assumption 1.1 holds. Then \( \overline{x}(0) \) is given by
\[
\overline{x}(0) = x(0)R_0(I - R)^{-1},
\]
where \( R = \sum_{k=0}^{\infty} R(k) \) and \( R_0 = \sum_{k=0}^{\infty} R_0(k) \). Furthermore, we have
\[
\pi = (1 - \rho)g(I - U(0))^{-1}(I - R)^{-1}.
\]
For \( n \in \mathbb{N} \), let \( \{R^{n*}(k); k \in \mathbb{Z}_+\} \) denote the \( n \)th-fold convolution of \( \{R(k); k \in \mathbb{Z}_+\} \) with itself, i.e., \( R^{1*}(k) = R(k) \) \((k \in \mathbb{Z}_+)\) and for \( n = 2, 3, \ldots \),
\[
R^{n*}(k) = \sum_{l=0}^{k} R^{(n-1)*}(k - l)R(l), \quad k \in \mathbb{Z}_+.
\]
Let $R^0(0) = I$ and $R^0(k) = O$ for all $k \in \mathbb{N}$. Furthermore, let $F(k)$ ($k \in \mathbb{Z}_+$) denote
\[ F(k) = \sum_{n=0}^{\infty} R^*(k). \]
It then follows from (5) that
\[ x(k) = x(0) R^0 \ast F(k), \quad k \in \mathbb{N}, \]
where $R^0 \ast F(k) = \sum_{l=0}^{k} R^0(k-l) F(l)$ for $k \in \mathbb{Z}_+$ (Takine 2004, Corollary 1).

Thus $x(k)$ is given by
\[ x(k) = x(0) R^0 \ast F(k), \quad k \in \mathbb{Z}_+. \] (8)

Let $a(z) = \det(I - z^{-1} \hat{A}(z))$, where $\hat{A}(z) = \sum_{k=0}^{\infty} z^k A(k)$. Since $\hat{A}(1) = A$ is stochastic, $a(1) = 0$. Let $\tau$ denote
\[ \tau = \max\{n \in \mathbb{M}; a(e^{i2\pi l/n}) = 0, l = 0, 1, \ldots, n-1\}, \] (9)
where $i = \sqrt{-1}$.

**Remark 2.1** It is known that $\tau$ is equivalent to the period of a Markov additive process $\{(Z_n, J_n); n \in \mathbb{Z}_+\}$ on $\mathbb{M} \times \mathbb{Z}$ such that for any $i, j \in \mathbb{M}$ and $k, l \in \mathbb{Z},$
\[ P(Z_{n+1} = k + l, J_{n+1} = j \mid Z_n = l, J_n = i) = [A(k+1)]_{i,j}, \]
where $A(k) = O$ for $k = -1, -2, \ldots$. By definition,
\[ \tau = \gcd\{k + q(i) - q(j); [A(k+1)]_{i,j} > 0, k \in \mathbb{Z}, i, j \in \mathbb{M}\}, \] (10)
where $q$ is some function from $\mathbb{M}$ to $\{0, 1, \ldots, \tau - 1\}$ (see, e.g., Appendix B in Kimura et al. 2010). Although Lemma B.2 in Kimura et al. (2010) states that function $q$ satisfying (10) is injective, this is not true, in general.

**Proposition 2.3** If Assumption 1.1 holds, then the period of $G_\bullet$ in (2) is equal to $\tau$.

**Proof.** Let $\tau_G$ denote the period of $G_\bullet$. Since $G_\bullet$ is irreducible and stochastic, it follows (see, e.g., Seneta 2006, Theorem 1.7) that
\[ \tau_G = \max\{n \in \mathbb{M}; \det(e^{i2\pi l/n} I - G_\bullet) = 0, l = 0, 1, \ldots, n-1\}. \] (11)
In what follows, we prove $\tau_G = \tau$.

Let $\hat{R}(z) = \sum_{k=0}^{\infty} z^k R(k)$. It then follows from Theorem 14 in Zhao et al. (2003) that

$$ I - z^{-1} \hat{A}(z) = z^{-1}(I - \hat{R}(z))(I - U(0))(zI - G), \quad 0 < |z| \leq 1. $$

Recall that under Assumption 1.1, the Markov chain $\{(X_n, S_n)\}$ is irreducible and positive recurrent. Thus $I - U(0)$ is nonsingular, and Corollary 30 in Zhao et al. (2003) shows that $\det(I - \hat{R}(z)) \neq 0$ for any complex number $z$ such that $0 < |z| \leq 1$. Therefore

$$ a(z) = 0 \text{ if and only if } \det(zI - G) = 0, \quad 0 < |z| \leq 1. $$

Note here that $\det(zI - G) = \det(zI - G_\bullet) \det(zI - G_T)$, and that $\det(zI - G_T) \neq 0$ for $|z| > 0$ because $G_T$ is a nilpotent matrix. As a result,

$$ a(z) = 0 \text{ if and only if } \det(zI - G_\bullet) = 0, \quad 0 < |z| \leq 1, $$

from which, (9) and (11) we have $\tau_G = \tau$. 

\begin{remark}
Proposition 2.3 implies that under Assumption 1.1, $\lim_{m \to \infty} G^m = \text{eg}$ if and only if $\tau = 1$.
\end{remark}

Along the lines of the proof of Theorem 4 in Takine (2004), we can readily prove the following proposition.

\begin{proposition}
Suppose Assumption 1.1 holds and there exists some random variable $Y$ in $\mathbb{Z}_+$ with finite positive mean such that $Y_e \in \mathcal{S}$ and the following limits exist.

$$ \lim_{k \to \infty} \frac{\overline{A}(k)}{\mathbb{P}(Y > k)} = \frac{C_A}{\mathbb{E}[Y]}, \quad \lim_{k \to \infty} \frac{\overline{B}(k)}{\mathbb{P}(Y > k)} = \frac{C_B}{\mathbb{E}[Y]}, $$

where $C_A$ and $C_B$ are nonnegative matrices satisfying $C_A \neq 0$ or $C_B \neq 0$. If $\tau = 1$, we then have

$$ \lim_{k \to \infty} \frac{\overline{\pi}(k)}{\mathbb{P}(Y_e > k)} = \frac{x(0)C_B e + \overline{\pi}(0)C_A e}{1 - \rho} \cdot \pi. $$

\end{proposition}

\begin{remark}
Theorem 4 in Takine (2004) asserts that (13) holds without $\tau = 1$, though the proof of the theorem requires that $\lim_{m \to \infty} G^m = \text{eg}$, i.e., $\tau = 1$ (see Takine 2003, Appendix A.6).
\end{remark}
3. Main Results

This section presents the main results of this paper. Since the period of \( G \) in (2) is equal to \( \tau \) (see Proposition 2.3), \( G \) can be partitioned as follows:

\[
G = \begin{pmatrix}
M_{\cdot,1} & M_{\cdot,2} & M_{\cdot,3} & \cdots & M_{\cdot,\tau} & M_T \\
O & G_{1,2} & O & \cdots & O & O \\
O & O & G_{2,3} & \cdots & O & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & G_{\tau-1,\tau} & O \\
G_{\tau,1} & O & O & \cdots & O & O \\
G_{T,1} & G_{T,2} & G_{T,3} & \cdots & G_{T,\tau} & G_T
\end{pmatrix},
\]

where \( M_{\cdot,\nu}'s (\nu = 1, 2, \ldots, \tau) \) are disjoint subsets of \( M_{\cdot} \) such that \( \bigcup_{\nu=1}^{\tau} M_{\cdot,\nu} = M_{\cdot} \). Let \( G^{(\nu)}_{\cdot} (\nu = 1, 2, \ldots, \tau) \) denote

\[
G^{(\nu)}_{\cdot} = G_{\nu+1,\nu+2} \cdots G_{\nu+\tau-1,\nu+\tau},
\]

where \( G_{\nu+\tau+1} = G_{\tau,1} \) and \( G_{T,\nu+\tau+1} = G_{l,l+1} \) for \( l = 1, 2, \ldots, \tau - 1 \). We then have

\[
G^{\tau} = \begin{pmatrix}
G^{(\tau)}_{1} & O & \cdots & O & O & O \\
O & G^{(\tau)}_{2} & \cdots & O & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
O & O & \cdots & G^{(\tau)}_{\tau-1} & O & O \\
O & O & \cdots & O & G^{(\tau)}_{T} & O \\
G^{(\tau)}_{T,1} & G^{(\tau)}_{T,2} & \cdots & G^{(\tau)}_{T,\tau-1} & G^{(\tau)}_{T,\tau} & (G_T)^T
\end{pmatrix},
\]

where \( G^{(\tau)}_{T,\nu} (\nu = 1, 2, \ldots, \tau) \) is a nonnegative \(|M_T| \times |M_{\cdot,\nu}| \) matrix such that \( \{\sum_{\nu=1}^{\tau} G^{(\tau)}_{T,\nu} + (G_T)^T\} e = e \). Note that \( gG^{\tau} = g \) and \( G^{(\tau)}_{\nu} (\nu = 1, 2, \ldots, \tau) \) is aperiodic, irreducible and stochastic. Let \( \tilde{g}_{\nu} = g_{\nu}/(g_{\nu}e_{\nu}) > 0 (\nu = 1, 2, \ldots, \tau) \), where \( g_{\nu} \) is a subvector of \( g \) corresponding to \( M_{\cdot,\nu} \) and \( e_{\nu} = e_{\nu[I_{M_{\cdot,\nu}}]} \). It then follows that \( \tilde{g}_{\nu} \) is a unique stationary probability vector of \( G^{(\tau)}_{\nu} \). Furthermore, since \( \lim_{n \to \infty} (G^{(\tau)}_{\nu})^n = e_{\nu} \tilde{g}_{\nu} (\nu = 1, 2, \ldots, \tau) \) and
\[ \lim_{n \to \infty} (G_T)^n = O, \] we obtain

\[ \lim_{n \to \infty} G^{nt} = \begin{pmatrix} e_1 \tilde{g}_1 & O & \cdots & O & O & O \\ O & e_2 \tilde{g}_2 & \cdots & O & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & \cdots & e_{\tau-1} \tilde{g}_{\tau-1} & O & O \\ O & O & \cdots & O & e_{\tau} \tilde{g}_{\tau} & O \\ f_1 \tilde{g}_1 & f_2 \tilde{g}_2 & \cdots & f_{\tau-1} \tilde{g}_{\tau-1} & f_{\tau} \tilde{g}_{\tau} & O \end{pmatrix}, \]

where \( f_\nu = [I - (G_T)^\tau]^{-1} G_{T,\nu}^\tau e_\nu (\nu = 1, 2, \ldots, \tau) \) and \( \sum_{\nu=1}^\tau f_\nu = e. \) It is easy to see that

\[ \lim_{n \to \infty} G^{nt} = E \Gamma, \] (14)

where \( E \) and \( \Gamma \) denote \( M \times \tau \) and \( \tau \times M \) matrices, respectively, such that

\[ E = \begin{pmatrix} e_1 & 0 & \cdots & 0 & 0 \\ 0 & e_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{\tau-1} & 0 \\ 0 & 0 & \cdots & 0 & e_{\tau} \\ f_1 & f_2 & \cdots & f_{\tau-1} & f_{\tau} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \tilde{g}_1 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{g}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{g}_{\tau-1} & 0 \\ 0 & 0 & \cdots & 0 & \tilde{g}_{\tau} \end{pmatrix}. \]

Note here that if \( M_T = 0 \), the corresponding rows and columns of \( E \) and \( \Gamma \) vanish. Note also that \( \Gamma \) satisfies

\[ \Gamma G^\tau = \Gamma, \quad \Gamma e = e^{[r]}, \] (15)

We now make an assumption and then show two lemmas, which will be used later to prove our main theorem.

**Assumption 3.1** There exists some random variable \( Y \) in \( \mathbb{Z}_+ \) with finite positive mean such that

\[ \lim_{k \to \infty} \frac{\overline{A}(k) E}{\mathbb{P}(Y > k)} = \frac{C^E_A}{E[Y]}, \quad \lim_{k \to \infty} \frac{\overline{B}(k) E}{\mathbb{P}(Y > k)} = \frac{C^E_B}{E[Y]}, \] (16)

where \( C^E_A \) and \( C^E_B \) are nonnegative \( M \times \tau \) and \( M_0 \times \tau \) matrices, respectively, satisfying \( C^E_A e^{[r]} \neq 0 \) or \( C^E_B e^{[r]} \neq 0. \)
Remark 3.1 If there exist some nonnegative matrices $C_A$ and $C_B$ satisfying (12), then Assumption 3.1 holds. Furthermore, if Assumption 3.1 holds, then (16) and $Ee^{[r]} = e^{[M]}$ yield
\[
\lim_{k \to \infty} \frac{A(k)e}{P(Y > k)} = \frac{C_A^Ee^{[r]}}{E[Y]}, \quad \lim_{k \to \infty} \frac{B(k)e}{P(Y > k)} = \frac{C_B^Ee^{[r]}}{E[Y]}.
\] (17)

Lemma 3.1 Suppose Assumptions 1.1 and 3.1 hold. If $Y_e$ are long-tailed (i.e., $Y_e \in \mathcal{L}$; see Definition A.1), then
\[
\lim_{k \to \infty} \frac{R(k)}{P(Y_e > k)} = \frac{C_A^Ee^{[r]}g(I - U(0))^{-1}}{(I - R)} - 1 \quad \text{(18)}
\]
\[
\lim_{k \to \infty} \frac{R_0(k)}{P(Y_e > k)} = \frac{C_B^Ee^{[r]}g(I - U(0))^{-1}}{(I - R)} - 1 \quad \text{(19)}
\]

Proof. See Appendix B.1. \qed

Lemma 3.2 Suppose Assumptions 1.1 and 3.1 hold. If $Y_e \in \mathcal{S}$, then
\[
\lim_{k \to \infty} \frac{F(k)}{P(Y_e > k)} = (I - R)^{-1}C_A^Ee^{[r]} \frac{\pi}{1 - \rho}. \quad \text{(20)}
\]

Proof. Since $Y_e \in \mathcal{S}$ and $\sum_{k=0}^{\infty} F(k) = (I - R)^{-1}$, it follows from (18) and Lemma 6 in Jelenković and Lazar (1998) that
\[
\lim_{k \to \infty} \frac{F(k)}{P(Y_e > k)} = \lim_{k \to \infty} \sum_{n=0}^{\infty} \frac{R^k(k)}{P(Y_e > k)} = (I - R)^{-1}C_A^Ee^{[r]}g(I - U(0))^{-1}(I - R)^{-1}.
\]
Substituting (7) into the above equation, we have (20). \qed

The following is our main theorem.

Theorem 3.1 Suppose Assumptions 1.1 and 3.1 hold. If $Y_e \in \mathcal{S}$, then
\[
\lim_{k \to \infty} \frac{x(k)}{P(Y_e > k)} = \frac{x(0)C_B^Ee^{[r]} + \bar{x}(0)C_A^Ee^{[r]}}{1 - \rho} \cdot \pi.
\]
Proof. Applying Proposition A.3 to (8) and using (19) and (20), we obtain

\[
\lim_{k \to \infty} \frac{\mathbf{x}(k)}{\mathbb{P}(Y > k)} = \mathbf{x}(0) \left[ C^E_B \mathbf{e}^{[\tau]} g(I - U(0))^{-1} (I - R)^{-1} \right.
\]

\[
+ R_0(I - R)^{-1} C^E_A \mathbf{e}^{[\tau]} \frac{\pi}{1 - \rho} \left. \right]
\]

\[
= \left[ \mathbf{x}(0) C^E_B \mathbf{e}^{[\tau]} \cdot \frac{\pi}{1 - \rho} + \bar{\mathbf{x}}(0) C^E_A \mathbf{e}^{[\tau]} \cdot \frac{\pi}{1 - \rho} \right],
\]

where the last equality follows from (6) and (7). □

Example 3.1 We consider a discrete-time FIFO BMAP/D/1 queue fed by a BMAP \( \{D(k); k \in \mathbb{Z}_+\} \) with \( M \) phases. We assume that service times are equal to a unit time and that \( \{D(k)\} \) satisfies

\[
D(2k) = \begin{pmatrix} O & D_{1,2}(k) \\ D_{2,1}(k) & O \end{pmatrix},
\]

\[
D(2k+1) = \begin{pmatrix} D_{1,1}(k) & O \\ O & D_{2,2}(k) \end{pmatrix},
\]

where \( D_{i,j}(k) (i, j = 1, 2, k \in \mathbb{Z}_+) \) is some positive matrix such that \( \sum_{k=0}^{\infty}(D_{i,1}(k) + D_{i,2}(k)) \mathbf{e} = \mathbf{e} \) and \( \bar{D}_{i,j}(k) \mathbf{e} \sim k^{c_{i,j}/(2k)^{\theta}} \) for some \( c_{i,j} > 0 \) and \( \theta > 1 \). It then follows that the stationary queue length distribution is identical to the stationary distribution of an M/G/1-type Markov chain with \( C(0) = A(0) \) and \( A(k) = B(k) = D(k) \) for \( k \in \mathbb{Z}_+ \). We can also confirm that \( \tau = 2 \), and that \( E \) and \( G \) have the following expressions:

\[
E = \begin{pmatrix} \mathbf{e} & 0 \\ 0 & \mathbf{e} \end{pmatrix}, \quad G = \begin{pmatrix} O & G_{1,2} \\ G_{2,1} & O \end{pmatrix}.
\]

Furthermore,

\[
\lim_{k \to \infty} \frac{\mathbf{A}(k) \mathbf{E}}{\mathbb{P}(Y > k)} = \lim_{k \to \infty} \frac{\mathbf{B}(k) \mathbf{E}}{\mathbb{P}(Y > k)} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} > \mathbf{0},
\]

for some regularly-varying random variable \( Y \) such that \( \mathbb{P}(Y > k) \sim k^{-\theta} \). Therefore this example satisfies the assumptions of Theorem 3.1.
Remark 3.2 The two results corresponding to Theorem 3.1 are presented in Theorem 5.1 (a) in Li and Zhao (2005a) and Theorem 5.1 in Asmussen and Møller (1999). The first one requires that $C_B^e = O$ and $Y \in S^*$, which implies $Y \in S$ and $Y_e \in S$ (see Definition A.3 and Proposition A.2). The second one requires that (i) $Y$ is regularly varying with finite positive mean, or (ii) $Y \in S$ and $\alpha(x) \triangleq E[Y - x \mid Y > x]$ is eventually non-decreasing and for some $t > 1$,
$$
\liminf_{x \to \infty} \frac{\alpha(tx)}{\alpha(x)} > 1.
$$
Clearly under condition (i), $Y_e$ is also regularly varying (thus $Y_e \in S$). Furthermore, Corollary 2.5 in Goldie and Resnick (1988) implies that if condition (ii) holds, then $Y_e \in S$ and $Y_e$ is in the maximum domain of attraction of the Gumbel distribution (see, e.g., Embrechts et al. 1997, Section 3.3). As a result, the assumptions of these results in Li and Zhao (2005a) and Asmussen and Møller (1999) are more restrictive than those of Theorem 3.1.

In the rest of this section, we consider the case of $\tau = 1$. When $\tau = 1$, $\Gamma = g$ and $E = e^{[M]}$. Therefore in this case, Assumption 3.1 is rewritten as follows.

Assumption 3.2 There exists some random variable $Y$ in $\mathbb{Z}_+$ with finite positive mean such that

$$
\lim_{k \to \infty} \frac{\mathcal{A}(k)e}{P(Y > k)} = c_A E[Y], \quad \lim_{k \to \infty} \frac{\mathcal{B}(k)e}{P(Y > k)} = c_B E[Y],
$$

where $c_A$ and $c_B$ are nonnegative vectors satisfying $c_A \neq 0$ or $c_B \neq 0$.

Corollary 3.1 Suppose Assumptions 1.1 and 3.2 hold. If $\tau = 1$ and $Y_e \in S$,

$$
\lim_{k \to \infty} \frac{\mathcal{X}(k)}{P(Y_e > k)} = \frac{\mathcal{X}(0)c_B + \mathcal{X}(0)c_A}{1 - \rho} \cdot \pi. \quad (21)
$$

We compare Corollary 3.1 with Proposition 2.4. We first suppose the assumptions of Proposition 2.4 are satisfied. Then (12) yields

$$
\lim_{k \to \infty} \frac{\mathcal{A}(k)e}{P(Y > k)} = c_A E[Y], \quad \lim_{k \to \infty} \frac{\mathcal{B}(k)e}{P(Y > k)} = c_B E[Y],
$$

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which shows that the assumptions of Corollary 3.1 hold for \( c_A = C_A e \) and \( c_B = C_B e \). On the other hand, the assumptions of Corollary 3.1 do not necessarily imply those of Proposition 2.4. To see this, we assume that \( \{ A(k) \} \) and \( \{ B(k) \} \) are of \( 2 \times 2 \) dimension and they satisfy \( \lim_{k \to \infty} A(k)/e^{-\sqrt{k}} = O \) and

\[
\begin{align*}
B_{1,1}(k) &\sim e^{-\sqrt{k} - \sin \sqrt{k} - 1}, \\
B_{1,2}(k) &\sim 2e^{-\sqrt{k}}, \\
B_{2,1}(k) &\sim 2e^{-\sqrt{k}}, \\
B_{2,2}(k) &\sim e^{-\sqrt{k}},
\end{align*}
\]

where \( B_{i,j}(k) \) denotes the \((i, j)\)th element of \( \overline{B}(k) \). Furthermore, we choose a random variable \( Y \) in \( Z_+ \) satisfying \( P(Y > k) \uparrow e^{-\sqrt{k}} \). Then \( Y_e \in S \) and

\[
\lim_{k \to \infty} \frac{A(k)e}{P(Y > k)} = 0, \quad \lim_{k \to \infty} \frac{B(k)e}{P(Y > k)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},
\]

which shows that the assumptions of Corollary 3.1 are satisfied. However in this example, there exists no random variable \( Y \) in \( Z_+ \) satisfying (12) in Proposition 2.4. Therefore, Corollary 3.1 is more general than Proposition 2.4.

The following is a special case, which can be applied to the stationary queue length distribution in the FIFO BMAP/GI/1 queue (see Masuyama et al. 2009; Takine 2000).

**Corollary 3.2** Suppose Assumptions 1.1 and 3.2 hold. If \( \tau = 1, Y_e \in S, C(0) = A(0) \) and \( B(k) = A(k) \) (\( \forall k \in Z_+ \)), then

\[
\lim_{k \to \infty} \frac{\pi(k)}{P(Y > k)} = \frac{\pi c_A}{1 - \rho} \cdot \pi. \tag{22}
\]

**Proof.** Clearly, \( c_A = c_B \) and \( \pi = \pi(0) + \pi(0) \). Thus (21) is reduced to (22). \( \square \)

Finally, we mention some asymptotic results given in Li and Zhao (2005a).

**Lemma 3.3** Suppose Assumption 1.1 holds and there exists some random variable \( Y \) in \( Z_+ \) with finite positive mean such that \( Y_e \in L \) and

\[
\lim_{k \to \infty} \frac{B(k)}{P(Y > k)} = V \neq O. \tag{23}
\]

Then \( \overline{R}_0(k)/P(Y > k) \) does not converge to any finite matrix as \( k \to \infty \).
Proof. See Appendix B.2. □

Actually, Theorem 4.1 in Li and Zhao (2005a) states that if Assumption 1.1 is satisfied and (23) holds for some $Y \in \mathcal{L}$ (thus $Y_e \in \mathcal{L}$), then

$$\lim_{k \to \infty} \frac{R_0(k)}{P(Y > k)} = V[(I - U(0))(I - G)]^{-1}. \quad (24)$$

However, the inverse matrix on the right hand side of (24) does not exist because $G$ is stochastic due to Assumption 1.1. By similar reasoning, we can see that the asymptotic equalities given in Corollary 5.1, Theorem 5.1 (b) and Theorem 5.2 in Li and Zhao (2005a) are incorrect. Furthermore, Theorem 5.1 (b) and Theorem 5.2 (ii) and (iii) in Li and Zhao (2005a) state that if the tail of $\{B(k)\}$ is equivalent to that of some $Y \in \mathcal{S}$ and also heavier than that of $\{A(k)\}$, then $\lim_{k \to \infty} \overline{\alpha}(k)/P(Y > k) = c$ for some finite $c \geq 0$.

The statement is, however, inconsistent with our Theorem 3.1 with $C^E_A = O$.

A. Heavy-Tailed Distributions

In this section, we describe some properties of heavy-tailed distributions, focusing on random variables in $\mathbb{Z}_+$. A random variable $X$ in $\mathbb{Z}_+$ and its distribution are said to be heavy-tailed if $E[z^X] = \infty$ for any $z > 1$. As is well known, there are two important subclasses of heavy-tailed distributions: the long-tailed class and the subexponential class.

Definition A.1 (Asmussen 2003; Sigman 1999) A random variable $X$ in $\mathbb{Z}_+$ and its distribution are said to be long-tailed if $P(X > k) > 0$ for all $k \in \mathbb{Z}_+$ and $P(X > k + 1) \approx P(X > k)$. The class of long-tailed distributions is denoted by $\mathcal{L}$.

Proposition A.1 If $X_e \in \mathcal{L}$, then for $h \in \mathbb{N}$, $l_0 \in \mathbb{Z}_+$ and $\nu = 0, 1, \ldots, h-1$,

$$\frac{1}{E[X]} \lim_{k \to \infty} \sum_{l=0}^{\infty} P(X > k + lh + \nu) \frac{P(X_e > k)}{P(X_e > k)} = \frac{1}{h}. \quad (A.1)$$

Proof. It follows from Corollary 3.3 in Sigman (1999) that for any fixed (possibly negative) integer $i$,

$$\lim_{k \to \infty} \frac{P(X > k + l_0h + i)}{\sum_{j=0}^{h-1} \sum_{l=0}^{\infty} P(X > k + lh + j)} = \frac{1}{E[X]} \lim_{k \to \infty} \frac{P(X > k + l_0h + i)}{P(X_e > k + l_0h - 1)} = 0.$$
Thus there exists some \( j^* \in \{0, 1, \ldots, h - 1 \} \) such that
\[
\lim_{k \to \infty} \frac{P(X > k + l_0h + i)}{\sum_{l=0}^{\infty} P(X > k + lh + j^*)} = 0.
\] (A.2)

For \( j^* \leq \nu \leq h \), we have
\[
1 \geq \frac{\sum_{l=0}^{\infty} P(X > k + lh + \nu)}{\sum_{l=0}^{\infty} P(X > k + lh + j^*)} \geq 1 - \frac{P(X > k + l_0h + j^*)}{\sum_{l=0}^{\infty} P(X > k + lh + j^*)},
\]
from which and (A.2) it follows that
\[
\lim_{k \to \infty} \frac{\sum_{l=0}^{\infty} P(X > k + lh + \nu)}{\sum_{l=0}^{\infty} P(X > k + lh + j^*)} = 1.
\] (A.3)

Similarly, (A.3) holds for \( 0 \leq \nu < j^* \). Finally, (A.3) yields
\[
\frac{1}{\mathbb{E}[X]} \lim_{k \to \infty} \frac{\sum_{l=0}^{\infty} P(X > k + lh + \nu)}{P(X_e > k + l_0h - 1)} = \lim_{k \to \infty} \frac{\sum_{l=0}^{\infty} P(X > k + lh + \nu)}{\sum_{m=0}^{\infty} P(X > k + m)} = \lim_{k \to \infty} \frac{\sum_{l=0}^{h-1} \sum_{i=0}^{\infty} P(X > k + lh + j) \cdot \sum_{l=0}^{\infty} P(X > k + lh + j^*)}{\sum_{l=0}^{\infty} P(X > k + lh + j^*)} = \frac{1}{h},
\]
which implies (A.1) because \( P(X_e > k + l_0h - 1) \overset{\sim}{\sim} P(X_e > k) \). \( \square \)

**Definition A.2 (Chistyakov 1964; Sigman 1999)** A random variable \( X \) in \( \mathbb{Z}_+ \) and its distribution are said to be subexponential if \( P(X > k) > 0 \) for all \( k \in \mathbb{Z}_+ \) and \( P(X_1 + \cdots + X_n > k) \overset{\sim}{\sim} n P(X > k) \) for all \( n = 2, 3, \ldots \), where \( X_i \)'s are independent copies of \( X \). The class of subexponential distributions is denoted by \( \mathcal{S} \).

The following is a discrete analog of class \( \mathcal{S}^* \), which is introduced by Klüppelberg (1988).
Definition A.3 A random variable \( X \) in \( Z_+ \) and its distribution belong to class \( S^* \) if \( P(X > k) > 0 \) for all \( k \in Z_+ \) and
\[
\lim_{k \to \infty} \sum_{l=0}^{k} \frac{P(X > k - l)P(X > l)}{P(X > k)} = 2E[X]. \tag{A.4}
\]

Proposition A.2 If a random variable \( X \) in \( Z_+ \) belongs to \( S^* \), then \( X \in S \) and \( X_e \in S \).

Proposition A.2 can be proved similarly to the proof of Theorem 3.2 (b) in Klüppelberg (1988). However for readers’ convenience, we provide a complete proof in Appendix B.3.

Proposition A.3 below characterizes the tail asymptotics of the convolution of two matrix sequences associated with a subexponential tail.

Proposition A.3 Let \( \mathbb{J}_n \) (\( n = 0, 1, 2 \)) denote a finite set. Let \( \{ P(k); k \in Z_+ \} \) and \( \{ Q(k); k \in Z_+ \} \) denote nonnegative \( |\mathbb{J}_0| \times |\mathbb{J}_1| \) and \( |\mathbb{J}_1| \times |\mathbb{J}_2| \) matrix sequences, respectively, such that \( P \triangleq \sum_{k=0}^{\infty} P(k) \) and \( Q \triangleq \sum_{k=0}^{\infty} Q(k) \) are finite. Suppose that for some random variable \( Y \in S \),
\[
\lim_{k \to \infty} P(k)P(Y > k) = \tilde{P} \geq O, \quad \lim_{k \to \infty} Q(k)P(Y > k) = \tilde{Q} \geq O,
\]
where \( \tilde{P} = \tilde{Q} = O \) is allowed. We then have
\[
\lim_{k \to \infty} \frac{P \ast Q(k)}{P(Y > k)} = \tilde{P}Q + P\tilde{Q}, \tag{A.5}
\]
where \( P \ast Q(k) = \sum_{l=0}^{k} P(k - l)Q(l) \) for \( k \in Z_+ \).

Proof. Let \( \mathbb{J}_1(i, j) = \{ \nu \in \mathbb{J}_1; [P]_{i, \nu}[Q]_{\nu, j} > 0 \} \) for \( i \in \mathbb{J}_0 \) and \( j \in \mathbb{J}_2 \). In what follows, we fix \( i \in \mathbb{J}_0 \) and \( j \in \mathbb{J}_2 \) arbitrarily. Let \( P(k) = \sum_{l=0}^{k} P(l) \) for \( k \in Z_+ \). Since \( P \ast Q(k) = PQ - \sum_{l=0}^{k} P(k - l)Q(l) \), we have
\[
[P \ast Q(k)]_{i, j} = \sum_{\nu \in \mathbb{J}_1(i, j)} \left( [P]_{i, \nu}[Q]_{\nu, j} - \sum_{l=0}^{k} [P(k - l)]_{i, \nu}[Q(l)]_{\nu, j} \right)
= \sum_{\nu \in \mathbb{J}_1(i, j)} [P]_{i, \nu}[Q]_{\nu, j} \left( 1 - \sum_{l=0}^{k} \frac{[P(k - l)]_{i, \nu}[Q(l)]_{\nu, j}}{[P]_{i, \nu}[Q]_{\nu, j}} \right)
= \sum_{\nu \in \mathbb{J}_1(i, j)} [P]_{i, \nu}[Q]_{\nu, j} P(P_{i, \nu} + Q_{\nu, j} > k), \tag{A.6}
\]
where $P_{i,\nu}$ and $Q_{\nu,j}$ ($\nu \in J_1(i,j)$) denote random variables in $\mathbb{Z}_+$ such that for all $k = 0, 1, \ldots$,

$$P(P_{i,\nu} = k) = \frac{[P(k)]_{i,\nu}}{[P]_{i,\nu}}, \quad P(Q_{\nu,j} = k) = \frac{[Q(k)]_{\nu,j}}{[Q]_{\nu,j}}.$$ 

Note here that for $\nu \in J_1(i,j)$,

$$\lim_{k \to \infty} P(P_{i,\nu} > k) = \lim_{k \to \infty} \frac{P(Y > k)}{[P]_{i,\nu}} = [\tilde{P}]_{i,\nu}.$$ 

Note also that if $[P]_{i,\nu} = 0$ (resp. $[Q]_{\nu,j} = 0$), then $[\tilde{P}]_{i,\nu} = 0$ (resp. $[\tilde{Q}]_{\nu,j} = 0$). Thus

$$[\tilde{P}]_{i,\nu}[Q]_{\nu,j} + [P]_{i,\nu}[\tilde{Q}]_{\nu,j} = 0, \quad \nu \in J_1 \setminus J_1(i,j). \quad (A.8)$$

By applying Lemma 10 in Jelenković and Lazar (1998) to (A.6) and using (A.7) and (A.8), we obtain

$$\lim_{k \to \infty} \frac{[P*Q(k)]_{i,j}}{[P]_{i,j}} = \sum_{\nu \in J_1(i,j)} [P]_{i,\nu}[Q]_{\nu,j} \left( [\tilde{P}]_{i,\nu} + [\tilde{Q}]_{\nu,j} \right) = \sum_{\nu \in J_1} \left( [\tilde{P}]_{i,\nu}[Q]_{\nu,j} + [P]_{i,\nu}[\tilde{Q}]_{\nu,j} \right),$$

which leads to (A.5). \hfill \Box

**B. Proofs**

**B.1. Proof of Lemma 3.1**

It follows from (3) and (4) that

$$\lim_{k \to \infty} \frac{R(k)}{P(Y > k)} = \lim_{k \to \infty} \frac{\sum_{m=0}^{\infty} A(k+m+1)G_m}{P(Y > k)} \cdot \frac{P(Y > k+1)}{P(Y > k)} (I - U(0))^{-1}. \quad (B.1)$$

$$\lim_{k \to \infty} \frac{R_0(k)}{P(Y > k)} = \lim_{k \to \infty} \sum_{m=0}^{\infty} \frac{B(k+m)G_m}{P(Y > k)} (I - U(0))^{-1}. \quad (B.2)$$

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It can be shown that
\[
\lim_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{\Pr(Y_e > k)} = C_A e^{\tau} g, \tag{B.3}
\]
\[
\lim_{k \to \infty} \sum_{m=0}^{\infty} \frac{B(k + m)G^m}{\Pr(Y_e > k)} = C_B e^{\tau} g. \tag{B.4}
\]

Applying (B.3) and \(\Pr(Y_e > k + 1) \sim \Pr(Y_e > k)\) to (B.1), we have (18). From (B.2) and (B.4), we also have (19). Therefore Lemma 3.1 is true.

In what follows, we provide the proof of (B.3), because (B.4) is proved in the same way. It follows from (14) and (16) that for any \(\varepsilon > 0\) there exists some positive integer \(m_* := m_*(\varepsilon)\) such that for all \(m > m_*\),
\[
(1 - \varepsilon)\bar{E} \Gamma \leq G^{[m/\tau]} \leq (1 + \varepsilon)\bar{E} \Gamma, \tag{B.5}
\]
\[
\frac{C_A e^{\tau} - \varepsilon e t}{E[Y]} \leq \frac{\bar{A}(m) E \Gamma}{\Pr(Y > m)} \leq \frac{C_A e^{\tau} + \varepsilon e t}{E[Y]}. \tag{B.6}
\]

We now have
\[
\limsup_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{\Pr(Y_e > k)} \leq \sum_{m=0}^{m_*} \limsup_{k \to \infty} \frac{A(k + m)G^m}{\Pr(Y > k + m)} \Pr(Y > k + m) \Pr(Y_e > k) \leq \sum_{m=m_*+1}^{\infty} \frac{A(k + m)G^m}{\Pr(Y_e > k)}. \tag{B.7}
\]

Note that \(G^m \leq e t^t \quad (\forall m = 0, 1, \ldots, m_*)\) and (17) yield
\[
\limsup_{k \to \infty} \frac{\bar{A}(k + m)G^m}{\Pr(Y > k + m)} \leq \frac{C_A e^{\tau} e^t}{E[Y]} < \infty,
\]
for \(m = 0, 1, \ldots, m_*\). Thus since \(Y_e \in \mathcal{L}\), the first term in (B.7) vanishes (see Corollary 3.3 in Sigman 1999) and therefore
\[
\limsup_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{\Pr(Y_e > k)} = \limsup_{k \to \infty} \sum_{m=m_*+1}^{\infty} \frac{A(k + m)G^m}{\Pr(Y_e > k)}
\]
\[
= \limsup_{k \to \infty} \sum_{\nu=0}^{\tau-1} \sum_{m \geq m_*+1}^{m \equiv \nu (\text{mod } \tau)} \frac{A(k + m)G^{[m/\tau]}G^\nu}{\Pr(Y_e > k)}. \tag{B.8}
\]
Substituting (B.5) into (B.8), we obtain
\[
\limsup_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{P(Y_e > k)} \leq (1 + \varepsilon) \sum_{\nu=0}^{\tau-1} \left[ \limsup_{k \to \infty} \sum_{m \geq m_\ast + 1}^{m \equiv \nu \ (\text{mod } \tau)} \frac{A(k + m)E\Gamma}{P(Y_e > k)} \right] G^\nu. \tag{B.9}
\]

It follows from (B.6) and Proposition A.1 that for any \( k \in \mathbb{Z}_+ \),
\[
\limsup_{k \to \infty} \sum_{m \geq m_\ast + 1}^{m \equiv \nu \ (\text{mod } \tau)} \frac{A(k + m)E\Gamma}{P(Y_e > k)} \leq \frac{C_A \Gamma + \varepsilon ee^t}{\tau} \limsup_{k \to \infty} \sum_{m \geq m_\ast + 1}^{m \equiv \nu \ (\text{mod } \tau)} \frac{P(Y > k + m)}{P(Y_e > k)} \tag{B.10}
\]
As a result, substituting (B.10) into (B.9) and letting \( \varepsilon \to 0 \) yield
\[
\limsup_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{P(Y_e > k)} \leq \frac{1}{\tau} C_A \Gamma \sum_{\nu=0}^{\tau-1} G^\nu. \tag{B.11}
\]
Similarly to the derivation of (B.11), we can obtain
\[
\liminf_{k \to \infty} \sum_{m=0}^{\infty} \frac{A(k + m)G^m}{P(Y_e > k)} \geq \frac{1}{\tau} C_A \Gamma \sum_{\nu=0}^{\tau-1} G^\nu. \tag{B.12}
\]
Note here that
\[
\Gamma \sum_{\nu=0}^{\tau-1} G^\nu = \Gamma (I - G^{-\tau} + \tau eG)(I - G + eG)^{-1} = \tau \cdot e^{[\tau]} g(I - G + eG)^{-1} = \tau \cdot e^{[\tau]} g,
\]
where the second equality follows from (15). Therefore, from (B.11) and (B.12), we have (B.3). \[ \square \]
B.2. Proof of Lemma 3.3

It follows from (4) and $(I - U(0))^{-1} \geq I$ that

$$\lim \inf_{k \to \infty} \frac{R_0(k)}{P(Y > k)} \geq \lim \inf_{k \to \infty} \sum_{m=0}^{\infty} \frac{B(k + m)G^m}{P(Y > k)}. \tag{B.13}$$

It follows from (14) and (23) that for any $\varepsilon > 0$ there exists some positive integer $m_0 := m_0(\varepsilon)$ such that for all $m > m_0$,

$$G_{\tau^\nu} \geq (1 - \varepsilon)E\Gamma, \quad \frac{B(m)}{P(Y > m)} \geq V - \varepsilon e \varepsilon. \tag{B.14}$$

Thus substituting (B.14) into (B.13), we have

$$\lim \inf_{k \to \infty} \frac{R_0(k)}{P(Y > k)} \geq \lim \inf_{k \to \infty} \sum_{\nu=0}^{\tau-1} \sum_{\substack{m \geq m_0 + 1 \\ m \equiv \nu \pmod{\tau}}} \frac{B(k + m)G_{\tau^\nu}G^\nu}{P(Y > k)} \geq (V - \varepsilon e \varepsilon)(1 - \varepsilon)E\Gamma \sum_{\nu=0}^{\tau-1} G^\nu \cdot \lim \inf_{k \to \infty} \sum_{\substack{m \geq m_0 + 1 \\ m \equiv \nu \pmod{\tau}}} \frac{P(Y > k + m)}{P(Y > k)}.$$

Proposition A.1 and Corollary 3.3 in Sigman (1999) yield

$$\lim \inf_{k \to \infty} \sum_{\substack{m \geq m_0 + 1 \\ m \equiv \nu \pmod{\tau}}} \frac{P(Y > k + m)}{P(Y > k)} = \infty.$$

As a result, the lemma is true. \qed

B.3. Proof of Proposition A.2

We first show $X \in \mathcal{L}$. For $k \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{2k} P(X > 2k - l)P(X > l) = 2 \sum_{l=0}^{k-1} P(X > 2k - l)P(X > l) + P(X > k)^2 \triangleq 2\varphi_1(k). \tag{B.15}$$
It then follows that for any fixed $\nu \in \mathbb{Z}_+$ and $k \geq \nu + 1$,

$$\varphi_1(k) = \sum_{l=0}^{k-1} P(X > 2k - l)P(X > l) + \frac{1}{2}P(X > k)^2$$

$$\geq \sum_{l=0}^{\nu-1} P(X > 2k - l)P(X > l) + \sum_{l=\nu}^{k-1} P(X > 2k - l)P(X > l)$$

$$\geq E[X]\left[ P(X > 2k)P(X_e \leq \nu - 1)
+ P(X > 2k - \nu)P(\nu - 1 < X_e \leq k - 1) \right],$$

which yields

$$1 \leq \frac{P(X > 2k - \nu)}{P(X > 2k)} \leq \left[ \frac{1}{E[X]} \frac{\varphi_1(k)}{P(X > 2k)} - P(X_e \leq \nu - 1) \right] \frac{P(\nu - 1 < X_e \leq k - 1)}.$$

(B.16)

Note here that (A.4) and (B.15) lead to

$$\lim_{k \to \infty} \varphi_1(k)/P(X > 2k) = E[X].$$

Since the right hand side of (B.16) converges to one as $k \to \infty$, we have $\lim_{k \to \infty} P(X > 2k - \nu)/P(X > 2k) = 1$. Similarly we can show that $\lim_{k \to \infty} P(X > 2k - 1 - \nu)/P(X > 2k - 1) = 1$ by using the following instead of (B.15).

$$\varphi_2(k) \triangleq \sum_{l=0}^{k-1} P(X > 2k - 1 - l)P(X > l) = \frac{1}{2} \sum_{l=0}^{2k-1} P(X > 2k - 1 - l)P(X > l)$$

As a result, we have $X \in \mathcal{L}$.

Next we show $X \in \mathcal{S}$. Note that for any positive integer $\eta < [k/2]$,

$$\sum_{l=0}^{k} \frac{P(X > k - l)P(X > l)}{P(X > k)} = 2 \sum_{l=0}^{\eta-1} \frac{P(X > k - l)P(X > l)}{P(X > k)} + \sum_{l=\eta}^{k-\eta} \frac{P(X > k - l)P(X > l)}{P(X > k)},$$

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from which, (A.4) and $X \in \mathcal{L}$ we have

$$
\lim_{\eta \to \infty} \limsup_{k \to \infty} \sum_{l=\eta}^{k-\eta} \frac{P(X > k - l)P(X > l)}{P(X > k)} = 0. \tag{B.17}
$$

Equation (B.17) and $\lim_{l \to \infty} \frac{P(X = l)}{P(X > l)} = 0$ (due to $X \in \mathcal{L}$) yield

$$
\lim_{\eta \to \infty} \limsup_{k \to \infty} \sum_{l=\eta}^{k-\eta} \frac{P(X > k - l)P(X = l)}{P(X > k)} = 0. \tag{B.18}
$$

Furthermore, it follows from $X \in \mathcal{L}$ that

$$
\lim_{\eta \to \infty} \lim_{k \to \infty} \sum_{l=0}^{\eta-1} \frac{P(X > k - l)P(X = l)}{P(X > k)} = 1, \tag{B.19}
$$

$$
\limsup_{k \to \infty} \sum_{l=k-\eta+1}^{k} \frac{P(X > k - l)P(X = l)}{P(X > k)} \leq \limsup_{k \to \infty} \frac{P(k - \eta < X \leq k)}{P(X > k)} = 0. \tag{B.20}
$$

From (B.18), (B.19) and (B.20), we have

$$
\lim_{k \to \infty} \sum_{l=0}^{k} \frac{P(X > k - l)P(X = l)}{P(X > k)} = 1,
$$

which implies $X \in \mathcal{S}$.

Finally, we show $X_e \in \mathcal{S}$. It follows from $X \in \mathcal{L}$ that $P(X_e = k + 1) \sim P(X_e = k)$. Further, from (A.4), we have

$$
\lim_{k \to \infty} \sum_{l=0}^{k} \frac{P(X_e = k - l)P(X_e = l)}{P(X_e = k)} = 2.
$$

Therefore, $X_e$ is locally subexponential, and thus $X_e \in \mathcal{S}$ (see Asmussen et al. 2003, Definition 2, Remarks 2 and 3). \qed
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